1 Introduction

Engineering design is most commonly achieved with the use of hard materials or simple fluids. The advantages of using soft, rheologically complex materials are demonstrated by many biological [1–3] and engineered [4–11] systems. A key challenge that inhibits the broader use of these materials in a systematic design is their complexity. The simplest rheological material properties are function-valued. These descriptors are known as “material functions” and can most generally depend on time or input amplitude (e.g., frequency dependent moduli, $G'(\omega)$, $G''(\omega)$, or shear thinning viscosity, $\eta(\dot{\gamma})$) in contrast to the constants necessary to describe simple solids or fluids.

Here, we focus on linear viscoelasticity (a subset of this complexity) for enhanced performance in mechanical systems. Viscoelastic and particularly linear viscoelastic material properties have been shown as important parameters governing the performance of real engineering systems including adhesives [6,8], vibration damping in skyscrapers [4,5], and automotive seatbelts and tires [12–14], and will be an integral part of “soft” matter systems such as soft robotics [7,15,16], and biomedical materials and devices [17–20].

Currently, much of the intersection between materials and design includes material selection [21] and material processing optimization [22], but there is an increasing interest in designing materials to achieve target functionality to meet a user’s need [23]. As described by Cussler and Moggridge [24], it is the functionality, not the molecule that produces the functionality that is important.

Rather than blindly trying to match existing or discovered materials to possible end use, the design process must begin with intended performance targets (such as vibration isolation, Refs. [4,5], pressure sensitive adhesive properties [6,8], top of Fig. 1). The process to design with complex materials discussed in this work will follow the top-down design hierarchy outlined in Fig. 1. The next step in design, specifically with rheologically complex materials, is to identify the rheological targets (material functions) to optimize this performance (middle of Fig. 1). In complex material design, this requires the use of governing equations of the system which involve system dependent features (i.e., geometry, type of deformation). The upward curved analysis arrow in Fig. 1 represents this necessity for proper mathematical design-appropriate modeling. For rheologically complex systems, this choice of modeling and design variables is nontrivial. Proper modeling is required to enable the inverse problem (downward curved design arrow) to identify the optimal function-valued properties. While it is common to design the geometry of a system, design with these rheological materials truly requires the cosedesign of both the material functions and the material geometry. Here, we isolate just the material property targets for design and optimization, made difficult enough by the function-valued nature of these targets.

The scope of the work presented in this paper lies in these early stages of the hierarchy to design with rheologically complex materials.
While the choice of design variables is obvious for simple materials (e.g., Young’s modulus for linear elastic solids), it is nontrivial for rheologically complex materials, including linear viscoelastic responses. Our criterion is that design-appropriate modeling should involve design variables that (i) encompass the most general material behavior, yet (ii) do not violate fundamental restrictions (e.g., Kramers–Kronig, see Sec. 2.1), and (iii) are directly measurable to facilitate development or selection of real materials. Not all viscoelastic models or material properties meet these criteria. Assuming a form for the viscoelastic response, such as a specific spring-dashpot arrangement or generalized Maxwell model or Prony series [33,34], does not incorporate power-law relaxation and thus fails criterion (i). Commonly used viscoelastic functions, such as complex moduli $G'(\omega)$ and $G''(\omega)$, are not independently designable and therefore fail criterion (i). A continuous relaxation spectrum response [27,28] may be useful; however, it is not directly measurable and does not meet criterion (iii). The relaxation modulus $G(t)$ (or its extrinsic equivalent, the relaxation kernel $K(t)$) and the creep-compliance modulus $J(t)$ both meet all the three criteria and are therefore design appropriate by our given criteria above. This work will focus on the former ($K(t)$ and $G(t)$) as the design-appropriate function-valued material property to use for target setting in the design of viscoelastic structures and materials.

As shown schematically in Fig. 2, a complex arrangement of linear springs and dashpots can be replaced by a generalized viscoelastic component whose behavior is governed by a single function-valued parameter $K(t)$ (mathematical details in Sec. 2.1). While the traditional spring-dashpot method is easy to conceptualize, it significantly limits the designer to a finite-dimensional design space. Although limited, the designer must also attempt to make systematic choices for both the topology of the connections and the spring/dashpot properties.

The introduction of a generalized viscoelastic connection expands the design space to include all possible discrete spring/dashpot arrangements in addition to behaviors that cannot be easily achieved with a typical system, such as power-law behavior which requires an infinite series of springs and dashpots. Topological arrangements of springs and dashpots as well as composite and heterogeneous materials can be designed to achieve target viscoelastic behavior [14,35]. Even further, viscoelastic materials themselves could potentially achieve the desired viscoelasticity $K(t)$.

Continuing from previous work [36], we develop a methodology to use function-valued properties, such as $K(t)$ in Fig. 2, as system-level variables while remaining material agnostic (i.e., agnostic to specific molecules or structure of the material). We use optimization methods to solve rheological material property target-setting problems. This methodology will be generalizable to the design of other classes of rheologically complex materials, where function-valued properties are known as material functions [37,38]. We demonstrate the utility of this method with an example of viscoelastic vibration attenuation.

Fig. 2 The introduction of a viscoelastic connection can change the design space from a discrete arrangement of linear springs and dashpots (left) to a single viscoelastic element with a relaxation kernel, $K(t)$.

Fig. 1 Function-valued viscoelastic properties may enable novel or improved performance (performance example schematics depict vibration damping of structures, vibration isolation of a suspended mass, and pressure sensitive adhesives). An early-stage design question is “what properties are optimal?” (represented here by the left, curved, downward arrow). This paper focuses on a methodology for answering that question with linear viscoelastic materials/systems including the nontrivial problem of selecting design-appropriate modeling using material properties (represented by the right, curved, upward arrow). Successful identification of targeted properties then leads to microstructure and formulation design (material selection or material synthesis) [25–28], which is beyond the scope of the work here.
2 Theory and Modeling

2.1 General Linear Viscoelastic Element. While previous efforts have addressed viscoelastic material design, some of these efforts have excluded the important feature of frequency-dependent storage and loss properties, resulting in extreme design limitations by ignoring the valuable ability of viscoelastic materials to adjust the properties with changes in loading frequency [39]. Other work has accounted for time-dependent behavior, but specifies that the linear viscoelastic relaxation function be described by a superposition of exponentials, known as a Prony series [34,40]. Although useful, this functional form limits the design space, since the structural form cannot include instantaneous damping (a delta function) and will require large numbers of parameters to describe simple functions such as power-law relaxation.

In its broadest form, the viscoelastic element shown in Fig. 2 can include any form of linear or nonlinear viscoelastic behavior, in which case the kernel function is input-amplitude dependent. To simplify the scope of design in this work, we consider the limit of linear viscoelasticity. In the linear viscoelastic limit, using Boltzmann superposition [38], the mechanical response of any material or structure is described by an experimentally measurable function that depends only on a timescale, the relaxation kernel, \( K(t) \). The entirety of \( K(t) \) can be optimized without regard to material class, spring-dashpot constitutive model, or ideally, parameterization of \( K(t) \).

Here, we consider one-dimensional deformation, and thus are able to represent the force through the viscoelastic connection with a single scalar equation

\[
F_{ve}(t) = \int_{-\infty}^{t} K(t-t')X(t')dt'
\]

where \( F_{ve} \) is the force due to a viscoelastic element, \( X \) is the deformation velocity experienced by the element (dimensions \( \dot{X} \equiv [LT^{-1}] \)) and the force relaxation kernel \( K(t-t') \equiv [FL^{-1}] \). With a change of variable \( s = t - t' \), this convolution integral becomes

\[
F_{ve}(t) = \int_{0}^{\infty} K(s)\dot{X}(t-s)ds
\]

If the viscoelastic properties arise from a continuum of material, we can make an analogy from extensive properties (force, velocity, stiffness) to intensive material properties (stress, strain, modulus), and therefore complex strain fields within the isotropic linear viscoelastic regime (see Table 1). The relaxation kernel \( K(t) \) that relates the extrinsic measures of force and displacement is transformed to its analogous material measures \( E(t) \) (in extension) or \( G(t) \) (in shear) which relate the intrinsic measures of stress and strain through a linear mapping of an area divided by length (dimensions \( [L] \)) based on geometry. A simple example of the relationship between the extrinsic and intrinsic measures is shown in one-dimensional tension where the intrinsic measures stress and strain are related through material connection function \( E \) (Table 1) by \( \sigma = E\epsilon \). Converting these measures to their extrinsic equivalents requires the geometrical relations \( \sigma = F/A \) and \( \epsilon = x/L \). Combining these relations leads to the extrinsic form of Hooke’s law, \( F = (EA/L)x \). Where the extrinsic force connection kernel \( K \) can be written as \( K = EA/L \equiv [F/L] \).

In the isotropic, incompressible, linear viscoelastic regime, the descriptive kernel function, the stress-relaxation modulus, is the measured stress response to a step change in strain, which can be used generally in Boltzmann superposition [7,38]. The modulus is equivalent to a predictive constitutive model parameter, and can be used to compute any three-dimensional deformation history. Within the linear viscoelastic limit, the mathematical formalisms for the relaxation kernel, \( K(t) \), discussed in this paper still apply to the relaxation modulus \( G(t) \). By analogy to Eq. (1), the 3D expression for the Boltzmann superposition integral uses the relaxation modulus \( G(t) \), in tensorial form

\[
\sigma(t) = \int_{-\infty}^{t} G(t-t')\dot{\gamma}(t')dt'
\]

where \( \sigma(t) \) is the Cauchy stress tensor and the strain-rate tensor is

\[
\dot{\gamma} = \sum \dot{\gamma} + (\sum \gamma)^T
\]

where \( \gamma \) is the velocity field and we define the velocity gradient as \( (\sum \gamma)_{ij} = \partial \gamma_{ij}/\partial x_i \). Equation (3) is limited to small deformation in the linear viscoelastic regime for incompressible materials, but applies for any class of linear viscoelastic material (elastomer, composite, polymeric liquid, colloid, gel, etc.) falling within the framework of a continuum description.

The convolution integral of Eq. (2) (or its material analogy Eq. (3)) presents a significant challenge when optimizing the functional form of \( K(t) \) (or \( G(t) \) in Eq. (3)). In particular, governing equations such as conservation of momentum cannot be written as instantaneous functions of state variables, but instead must involve an integral over all past time.

The relaxation kernel \( K(t) \) is treated here as an independent design variable. Alternative viscoelastic material functions, such as the creep compliance \( J(t) \), can also be used to define a relation similar to Eq. (3) with the displacement field as the output of the integral. This may be mathematically convenient for load-control inputs. However, in the linear viscoelastic limit, all of the material functions are interrelated [13], and therefore only one single-valued function can be specified in the design.

Linear viscoelastic materials are commonly described in the frequency domain, such as the dynamic storage and loss moduli (shown in Table 1). For a viscoelastic fluid (with stress relaxing to zero at infinite time), the dynamic storage and loss moduli are directly related to \( K(t) \) as

\[
K'(\omega) = \omega \int_{0}^{\infty} K(s)\sin(\omega s)ds
\]

\[
K''(\omega) = \omega \int_{0}^{\infty} K(s)\cos(\omega s)ds
\]

Table 1 Relaxation kernel, \( K(t) \), for the generalized viscoelastic component as shown in Fig. 2. For 1D systems, extrinsic component measures can be transformed to intrinsic material measures by a linear mapping depending on component geometry. The relaxation kernel is the design parameter as the dynamic storage and loss moduli are related through the Kramers–Kroenig relations, and thus not fully independent (Eqs. (7) and (8)).

<table>
<thead>
<tr>
<th>Force connection ( K \equiv [F/L] )</th>
<th>Material (extension) ( E \equiv [F/L]^2 )</th>
<th>Material (shear) ( G \equiv [F/L]^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relaxation kernel</td>
<td>( K(t) )</td>
<td>( K'(\omega) )</td>
</tr>
<tr>
<td>Dynamic storage</td>
<td>( K'(\omega) )</td>
<td>( E'(\omega) )</td>
</tr>
<tr>
<td>Dynamic loss</td>
<td>( K''(\omega) )</td>
<td>( E''(\omega) )</td>
</tr>
</tbody>
</table>

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Since each function is related to $K(t)$, the two functions are clearly not independent. As shown in Table 1, this holds true in the analogous material measures in extension ($E(t)$, $E'(\omega)$, $E''(\omega)$) and shear ($G(t)$, $G'(\omega)$, $G''(\omega)$). The interrelations are given by the Kramers–Kronig relations, shown here in terms of the dynamic moduli [38]

$$G'(\omega) - G'(\infty) = \frac{2}{\pi} \int_0^{\infty} \frac{G'(x)}{x^2 - \omega^2} \, dx$$

$$G''(\omega) = \frac{2\omega}{\pi} \int_0^{\infty} \frac{G'(x)}{x^2 - \omega^2} \, dx$$

where we define

$$G'(\infty) = \lim_{\omega \to \infty} G'(\omega)$$

Any in-phase and out-of-phase dynamic material functions must also satisfy these relations. With simple shear properties, this includes moduli $G'$ and $G''$, viscosities $\eta'$ and $\eta''$, compliances $J'$ and $J''$, and fluidities $\phi'$ and $\phi''$ [37,38]. The Kramers–Kronig relations also restrict the independent specification of frequency-dependent magnitude and phase angle.

The independent function $K(t)$ (or equivalently $G(t)$ or $E(t)$) is therefore treated as the function-valued design variable for linear viscoelasticity.

2.2 Parameters of $K(t)$. In general, the relaxation kernel, $K(t)$, can be treated as a function of arbitrary structure. Passive materials or systems impose the restriction that the function be monotonically decreasing [41,42]; however, this restriction on complexity could be lifted through the use of actively controlled systems.

Complete freedom in the shape of the relaxation kernel presents difficulties for numerical optimization. For this initial exploration of $K(t)$ optimization, we consider several parameterizations of $K(t)$ with a finite number of design parameters as shown in Fig. 3: parameterization as a dashpot, a single and multimode Maxwell, and power-law relaxation (analogous to a critical gel material) [43,44]. These parameterizations represent commonly used conventional models. They were chosen both for their mathematical and conceptual simplicity and also because real material systems are known to follow this basic behavior (see for Maxwell fluid [45–47] and for critical gel [43,44,48,49]).

For a standard, linear dashpot, the form of the relaxation kernel can be represented as

$$K(t) = c \cdot \delta(t)$$

where $c = |F/(L/T)|$. The dynamic coefficients for a linear dashpot are given by

$$K'(\omega) = 0$$

$$K''(\omega) = c \cdot \omega$$

By analogy, a Newtonian fluid has $G(t) = \eta_0 \delta(t)$, $G' = 0$, and $G'' = \eta_0 \omega$. A Maxwell element is a linear spring and dashpot connected in series. It is the simplest model of a viscoelastic fluid. The model can be generalized to a multimode Maxwell model that includes $M$ Maxwell elements connected in parallel. The relaxation kernel, also known as a Prony series, is defined by

$$K(t) = \sum_{n=1}^{M} K_m e^{-t/\lambda_m}$$

where $K_m$ are the Maxwell spring constants ($K_m = [F/L]$) and $\lambda_m$ are the relaxation times ($\lambda_m = [T]$). The Maxwell dashpot coefficient is $\eta_m = K_m \lambda_m$. Here we will consider the cases of $M = 1$ and $M = 3$ in order to limit the number of parameters ($2M$ parameters for an $M$-mode Maxwell model). For this form, the dynamic coefficients of a force component connection are

$$K'(\omega) = \sum_{n=1}^{M} K_m \frac{(\lambda_m \omega)^2}{1 + (\lambda_m \omega)^2}$$

$$K''(\omega) = \sum_{n=1}^{M} K_m \frac{\lambda_m \omega}{1 + (\lambda_m \omega)^2}$$

These are analogous to intrinsic material properties $G'(\omega)$, $G''(\omega)$ (see Table 1). To achieve some functional forms of $K(t)$ such as power-law behavior, a Maxwell model would require a large number of parameters. In order to include this broader design space, but with a small number of parameters, we will consider the critical gel power-law model. Power-law rheology is an important signature, seen in materials near a gel point [44,50], food systems such as egg yolk [48], and active biological materials such as cells [51]. Present literature describes this behavior in terms of intrinsic material characteristics $G(t)$, $G'(\omega)$, $G''(\omega)$ [43,44] which we use to generate their force connection analogs. In component design, these measures are related to $K(t)$, $K'(\omega)$, and $K''(\omega)$ by geometric factors. The critical gel behavior is described by the power-law equation

$$K(t) = S_i t^{-n}$$

where $S_i$, dimensions $[F(L) \cdot T^n]$, is the gel strength parameter and $n$ is the power-law coefficient. The exponent is typically $n \approx 1/2$, but more generally is restricted to the range $0 < n < 1$. The dynamic moduli for the critical gel can be generalized to an extrinsic force connection as [43,44,50].
\[ K'(\omega) = \frac{K''(\omega)}{\tan\left(\frac{n\pi}{2}\right)} = \Gamma(1-n)\cos\left(\frac{n\pi}{2}\right) S_n e^{\omega^n} \] (17)

where \( \Gamma(\cdot) \) is the Gamma function. The first equality of Eq. (17) links stiffness and damping, further constraining viscoelastic properties differently than assumed in Sec. 2.1. Note that \( K'(\omega) \) and \( K''(\omega) \) both scale as \( \omega^n \) in a log-log plot (i.e., the modulus lines are parallel). Importantly, there is no simple mechanical analogy with springs and dashpots for the critical gel (it requires an infinite set of parallel Maxwell elements). Yet, this behavior is accessible with real materials, if not (overly simplified) spring and dashpot models.

While some parameterizations (e.g., Maxwell model) imply the use of a specified topology of linear springs and dashpots, this is not generally necessary for a viscoelastic connection. One could also parameterize \( K(t) \) as a spline, or a discrete vector of independent points. This would be the ideal approach with maximum design freedom. A key difficulty is solving convolution integrals with high degree of freedom parameterizations. This challenge is beyond the scope of the work here which is to simply demonstrate the concept of velocities. This is an important challenge to be addressed for general design of \( K(t) \) or \( G(t) \), and is the subject of ongoing work. Here, we simplify the analysis by considering time-periodic solutions, for which the convolution integral simplifies.

In the most general case, the optimization problem considered here seeks to minimize an objective function that describes the performance of the overall system and depends on the choice of \( K(t) \). Since the optimization is performed with respect to \( K(t) \), which is a function-valued design variable, this problem would fall under the class of optimal control problems [32]. While there are well-established methods to solve these types of problems [53,54], the structure arising from the problem is complex due to the characteristic convolution integrals and is not yet well understood. Thus, for the purpose of this paper, we demonstrate certain parameterizations of \( K(t) \) which allow for the much simpler and well established utilization of optimization algorithms in MATLAB for finite dimensional nonlinear programs.

3 Case Study

In general the linear viscoelastic element of Sec. 2.1 could be used to connect any two pieces in a system. As a case study, we consider a simple vibration isolator, shown in Fig. 4. This toy model is a one-dimensional abstraction of more sophisticated vibration isolation systems, such as those found in buildings or automotive systems [4,5,55]. In the initial case (Fig. 4(a)), a mass \( m \) is connected by a spring to a base that is moving with a prescribed displacement of \( y(t) \). The objective is to isolate the mass from the base displacement. A simple improvement is the addition of a linear dashpot (Fig. 4(b)). We will generalize the linear dashpot to be a parallel viscoelastic connection with relaxation kernel \( K(t) \), described in Sec. 2.1. We will demonstrate the added performance from a viscoelastic connection, and optimize \( K(t) \) based on the parameterizations described in Sec. 2.2.

3.1 Governing Equations. Given an initial condition \( F_e(t = 0) = 0 \), Eq. (2) has limits of integration from 0 to \( t \). In general, the expression requires convolution of the kernel function \( K(t) \) with the entire time-history of the velocity experienced by the element, \( y(t) \).

For the particular system in Fig. 4, with a generalized viscoelastic element and the initial condition \( F_e(t = 0) = 0 \), the governing equation for conservation of linear momentum is most generally written as

\[-k(x - y) - \int_0^t K(s)[\dot{x}(t - s) - \dot{y}(t - s)]ds = m\ddot{x}(t) \] (18)

where \( s = t - \tau \) and \( x - \dot{y} = \dot{X} \), or the velocity of deformation, as defined in Eq. (2).

The convolution integral structure has two important consequences. First, the equations cannot be written in matrix form. Second, the numerical simulation of this model requires increased computation at each time step, since each time derivative function evaluation requires an integration of the entire prior time-history of velocities. This is an important challenge to be addressed for general design of \( K(t) \) (or \( G(t) \)), and is the subject of ongoing work. Here, we simplify the analysis by considering time-periodic solutions, for which the convolution integral simplifies.

Using complex notation, the displacement of the mass has the form

\[ x(t) = \text{Im}\{x^* e^{iu \omega t}\} \] (19)

where \( \text{Im}\{\} \) takes the imaginary portion of the complex quantity. The coefficients are

\[ x^* = X_R + iX_i \] (20)

By substituting Eqs. (19) and (20) into Eq. (18), a linear system of two equations and two unknowns will result. The system of equations takes the form

\[ Mx = B \] (21)

The unknowns are

\[ x = [X_R, X_i]^T \] (22)

The nonhomogeneous portion is

\[ B = [-Y(\omega S + k), -Y\omega C]^T \] (23)

![Fig. 4 Design of optimal viscoelastic vibration isolation for a one-dimensional spring-mass system (a); (b) the typical approach of an arrangement of springs and dashpots, and (c) a generalized viscoelastic element with relaxation kernel, \( K(t) \). The latter approach increases design freedom and identifies more optimal targets.](http://mechanicaldesign.asmedigitalcollection.asme.org/content/mechanicaldesign/vol.138/051402-5)

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and the $2 \times 2$ matrix is given by

$$
M = \begin{bmatrix}
(m\omega^2 - \omega S - k) & (\omega C) \\
(-\omega C) & (m\omega^2 - \omega S - k)
\end{bmatrix}
$$

(24)

The scalar coefficients $C$ and $S$ require integral calculations that depend on the kernel function $K(t)$

$$
C(\omega) = \int_0^\infty K(s)\cos(\omega s)ds
$$

(25)

$$
S(\omega) = \int_0^\infty K(s)\sin(\omega s)ds
$$

(26)

Comparing this result with Eqs. (5) and (6) shows that these integrals are related to the dynamic material functions as

$$
C = K''(\omega)/\omega
$$

(27)

$$
S = K'(\omega)/\omega
$$

(28)

The primary design variable is still $K(t)$, since it gives the rheological signature of both dynamic moduli, and the dynamic moduli are not independent parameters due to the Kramers–Kronig relations in Eqs. (7) and (8).

The solutions to the governing equations give $|\dot{x}|$, the displacement amplitude of the mass, as a function of frequency. Normalizing this result by the input displacement amplitude, $Y$, leads to the nondimensionalized amplitude

$$
|\dot{x}| = \frac{|x|}{Y}
$$

(29)

From the displacement, the acceleration amplitude of the mass is defined to be

$$
|\ddot{x}| = \omega^2|x|
$$

(30)

Equivalently, it can be nondimensionalized by the problem inputs of displacement ($Y \equiv [L]$), mass ($m \equiv [M]$), and spring constant ($k \equiv [M]/[T]$) as

$$
|\ddot{x}| = \frac{|x|}{Y \left(\frac{k}{m}\right)}
$$

(31)

In general, $K(t)$ can take any form, but here $K(t)$ is parameterized using the methods described in Sec. 2.2. The response $|\ddot{x}|$ is optimized by minimizing the peak value with respect to a finite set of design variables that parameterize $K(t)$. These parameters can themselves be nondimensionalized as follows. For the linear dashpot, $\dot{c} = c/\sqrt{km}$; for the Maxwell model, $\dot{\lambda} = \lambda/\sqrt{m/k}$ and $\dot{\eta} = \eta/\sqrt{km}$; and for the critical gel, $S = S/\sqrt{kT}$. Note that the critical gel exponent, $n$, is dimensionless.

The objective function $f$ for the design optimization problem here is the maximum nondimensionalized acceleration amplitude

$$
f(x) = \max|\ddot{x}|
$$

(32)

3.2 Results. The typical approach for vibration isolation is to combine a linear spring and dashpot in parallel, as in Fig. 4(b). The addition of the dashpot is equivalent to the parameterization for the relaxation kernel given in Eq. (10). In this parameterization, the design space has only one dimension, the dimensionless damping coefficient defined as

$$
\zeta = c/2m\omega_n
$$

(33)

The effect of varying this parameter is shown in the displacement and acceleration amplitude responses in Fig. 5, where a nondimensional displacement amplitude ($|\ddot{x}| = |x|/Y$) and the nondimensional acceleration amplitude are plotted against a frequency normalized by the resonant frequency of the initial system, $\tilde{\omega} = \omega/\sqrt{k/m}$. The resonant frequency would be influenced by any elasticity in $K(t)$.

This presents the clearest design tradeoff in optimization. As the dissipation increases, it attenuates the resonant response in both the displacement and acceleration at the natural frequency of the system ($\omega_n = \sqrt{k/m}$). Any finite dissipation, however, changes the high-frequency displacement response of the system from a scaling $|x| \sim \omega^{-2}$ to $|x| \sim \omega^{-1}$. This leads to a penalty at high frequency, most clearly demonstrated with the acceleration $|\ddot{x}| \sim \omega^2$, of the mass. A finite amount of damping changes the slope of the high frequency acceleration behavior from $|\ddot{x}| \sim \omega^0$ to $|\ddot{x}| \sim \omega^1$. While damping can decrease the peak acceleration at the resonant frequency, it comes at the cost of ever increasing acceleration at large $\omega$ (Fig. 5(b)). (See Supplemental Information for the derivation of analytical scaling laws. This material is available under the “Supplemental Material” tab on the ASME digital collection.)
Relaxing the design space to include even simple parameterizations of viscoelastic fluids can improve the behavior of the vibration isolator, as demonstrated in Fig. 6. While parametrization does not provide infinite design freedom, it is valuable for this initial treatment, as it allows a visualization of the extended design space. A Maxwell model allows for the introduction of an additional timescale, \( \lambda \) from Eq. (13); for characteristic deformation timescales longer than \( \lambda \), the connection (or material) behavior transitions from spring-like (elastic) to dashpot-like (viscous). The high-frequency (short timescale) elastic behavior is a key feature. Properly designed, this allows for an attenuation of the peak acceleration at a resonant frequency without the penalty of ever-increasing acceleration at large \( \omega \). This is illustrated in Fig. 6. Note \( \omega = \omega / \omega_{\text{res}} \), where \( \omega_{\text{res}} \) depends on the Maxwell parameters. Other parameterizations likewise allow for reduction of the resonant acceleration with reduced penalty at high frequencies.

Both the critical gel and single mode Maxwell model expand the design space from the conventional linear dashpot solution. A portion of the expanded design space for these parameterizations is shown in Fig. 7.

We optimized the different forms of the kernel function to minimize the peak acceleration of the mass \( m \). Optimization was performed using the MATLAB \( \text{optimization toolbox with the functions "ga" and "fmincon" [57]. The results for the optimal viscoelastic kernel functions are presented in Table 2. The optimal values for a single mode Maxwell model and a critical gel model are shown as the white dots in Fig. 7.}

In the case of a multimode Maxwell model, the optimized viscoelastic parameters are not listed in Table 2, since there was too much variation in the individual values. However, all values for \( \eta \) and \( \lambda \) produced similar \( K(t) \) curves (in addition to being similar to the single mode Maxwell model). This shows that the shape of the curve is the key factor in optimization, not its exact parameterization.

The peak acceleration can occur either as a local maximum near resonance (e.g., Fig. 6, Maxwell), or at the highest frequency considered (e.g., Fig. 6, dashpot). Therefore, some solutions depend on the frequency range of interest. Shapes of all optimized \( K(t) \) curves are shown in Fig. 8. For both the dashpot and critical gel, the acceleration amplitude increases without bound at higher frequency, thus increasing the upper-bound of the frequency range will increase the acceleration amplitude at that value, possibly above the local peak value at the resonant frequency. For real applications, a finite range of frequency is reasonable, e.g.,

![Fig. 6 Design involving a generalized viscoelastic element with relaxation kernel \( K(\tau) \) can eliminate high-frequency acceleration. Here shown with Maxwell element (dashed–dotted line), and critical gel (short-dashed line) compared to dashpot (solid line), or no additional component (dashed line).](image)

![Fig. 7 The design space for (a) the single mode Maxwell element and (b) the critical gel to isolate vibrations in a single mass-spring system (given by Eqs. (13) and (16)). Each model increases the design space to two dimensions, compared to the single dimension of a simple linear dashpot. Higher saturation correlates with lower peak normalized accelerations, as given in the scale bar (right), under simple sinusoidal displacement forcing. The white circle identifies a global minimum of peak acceleration.](image)
excitation amplitudes decrease and may be negligible above a critical frequency.

Introducing a linear viscoelastic connection decreases the peak acceleration beyond that of a simple linear dashpot. With the introduction of a characteristic timescale, a Maxwell model provided the most improvement compared to the conventional method. A critical gel element also provides a marked improvement. Surprisingly, in this simplified problem, additional design freedom in the form of additional viscoelastic parameters (i.e., a multimode Maxwell model) does not provide further improvement over a single mode Maxwell model. This is likely due to the fact that this toy problem has only one characteristic frequency, thus only one additional time scale is necessary for optimal performance.

These results suggest that a critical gel model and multimode Maxwell model may further improve performance for more complex problems involving a more complicated forcing function. Critical gel performance may prove beneficial for excitations that superpose multiple frequencies, or for structures with multiple vibration modes.

### 3.3 Case Study Extension

As an extension to the above results and in order to get better intuition for more practical systems, the dynamic vibration isolator was further generalized to consider a range of nondimensional natural frequencies $\tilde{\omega}$ for which a single viscoelastic element is used. This natural frequency is normalized by some reference spring-mass system ($k_0, m_0$) such that

\[
\tilde{\omega} = \frac{\omega}{\omega_0}
\]

where $\omega_0$ is the natural frequency of the reference system $\omega_0 = \sqrt{k_0/m_0}$.

This is analogous to a “one size fits all” damper where the primary mass and spring may vary by application (Fig. 9). As shown before, use and optimization of a generalized viscoelastic element for a fixed natural frequency (i.e., $\tilde{\omega} = 1$), significantly improves the system response. However, the benefit of the optimized viscoelastic element is diluted for a system whose natural frequency is different from that for which the system is optimized.

In practice, it is unlikely that a system will be as idealized as the simple spring-mass isolator system proposed in the previous case study. A simple example can be thought of as a vehicle suspension system, where the vehicle is expected to perform over a wide range of additional mass. By extending the original example to include a range of natural frequencies, we allow for the possibility of added mass or a range of system stiffness.

For the previous studies optimized for a fixed $\tilde{\omega} = 1$, the objective function was chosen to be the maximum of the acceleration curve. Now, with a range of natural frequencies, selecting the absolute maximum would result in a discontinuous and non-smooth objective function. To avoid problems associated with discontinuities, the following (similar) objective function was used instead:

\[
f(x) = \left[ \sum_{i=1}^{m} \max(X_i) \right]^{1/n}
\]
where the nondimensional frequency increases and decreases by an order or magnitude, i.e.,

$$\frac{\omega}{C^3} \approx 10^{-1}$$

and $\omega^*$ refers to the corresponding acceleration versus frequency curve for each $\frac{\omega}{C^3}$.

Equation (35) is the $n$th norm of the peak acceleration for a set of natural frequencies. It becomes essential to assign a reasonable value for $n$ so that the objective function adequately weighs the cost of a high peak in acceleration. Here, $n$ is chosen to be 4. The results from this optimization formulation are discussed below.

The Maxwell model and critical gel model were both used within this new optimization scheme. The dashed lines in Fig. 10 show the system design (right) and performance (left) over a range $\frac{\omega}{C^3}$ optimized for a reference natural frequency ($\frac{\omega}{C^3} = 1$) and applied to the systems with a range of $\frac{\omega}{C^3}$.

**Fig. 9** The simple vibration isolator problem is extended to optimize the performance for a range of natural frequencies of the system but a fixed $K(t)$. Increasing the natural frequency of the system ($\omega_1 = \sqrt{k/m}$) can be conceptualized by increasing the spring constant $k$ and/or decreasing the system mass, $m$, shown in (a)–(c).

**Fig. 10** The performance (left) and viscoelastic (VE) design (right) for a Maxwell element and Critical gel element in the extended vibration isolation problem. Multimode and single mode system response was identical, thus only a single mode is shown. The system performance is measured by the acceleration response while the viscoelastic design corresponds to the kernel function, $K(t)$, of the viscoelastic element. Dashed lines represent optimization for a reference nondimensional natural frequency $\omega^* = 1$, where $\omega^*$ is as described in Eq. (34); solid lines represent the results optimized for the range of $\omega^*$. In the left panels (performance), the values of the objective function (fourth order norm of the maximum accelerations of each curve), optimized for a reference $\omega^* = 1$ (dashed lines) and a range of $\omega^*$ values are shown as horizontal lines. The optimizer works to minimize this objective function.
resonance acceleration \((\omega^* = 10 \text{ rad/s})\), leading to a design that is optimized in a more balanced manner, reducing the objective function (Eq. (35)) by 29%.

Likewise, the critical gel model optimized for the reference natural frequency \((\omega^* = 1)\) leads to problems with high frequency behavior for lower natural frequencies and resonance attenuation for higher natural frequencies. Optimizing for the entire range of natural frequencies allows for improved overall results for the system, as shown in the solid lines of Fig. 10. The objective function is decreased by 51%.

In both viscoelastic parameterizations, overall optimization of the system comes at a cost of higher maximum accelerations for some natural frequencies, shown as the circles along the nondimensional acceleration axes in Fig. 10. Values of nondimensionalized acceleration are shown in Table 3.

For the extension of the original case study, the maximum acceleration amplitude was reduced for both the single element Maxwell and critical gel models. As in the previous case study, the additional degrees of design freedom in the multimode Maxwell model provided no further improvement over a single mode. The full results of this extension are presented in Table 3.

The new optimization scheme dramatically affects the optimized kernel function shape, \(K(t)\). The kernel functions optimized for a fixed natural frequency and applied to systems with a range of natural frequencies and the kernel function optimized for a range of natural frequency are shown in Fig. 10. The expanded range pushes the optimal characteristic timescale, \(\tilde{\lambda}\), lower to improve the high-frequency behavior of the system. The range of frequencies decreases the magnitude of \(n\), the critical gel exponent, effectively making the system more solidlike, again improving the high-frequency behavior of the system.

An important note to make is that these results, even more so than the original problem, are dependent on the range of input frequency \((\omega)\) chosen. The ever-increasing acceleration at high frequency for a liquidlike system as well as a critical gel model drives the optimized results.

Passive vibration isolators find use in a variety of practical applications. In their use as air isolators in large industrial equipment, the typical natural frequency range is 1.5–3 Hz. In the case of base isolators used in buildings and large structures, the natural frequency varies over a low seismic range. Their usage in vehicles and aviation pertains to a frequency range of 10–20 Hz. The key takeaway here is that real world vibration isolation occurs over not just one but a range of resonant frequencies, for which viscoelastic responses can be optimized.

<table>
<thead>
<tr>
<th>Optimized for</th>
<th>Viscoelastic parameters</th>
<th>(\omega^* = 0.1)</th>
<th>(\omega^* = 1)</th>
<th>(\omega^* = 10)</th>
<th>Fourth norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maxwell</td>
<td>(\phi = 1)</td>
<td>(\tilde{\lambda} = 0.277)</td>
<td>(\eta = 0.553)</td>
<td>21.0</td>
<td>3.00</td>
</tr>
<tr>
<td>Range of (\omega^*)</td>
<td>(\tilde{\lambda} = 0.0952)</td>
<td>(\eta = 0.137)</td>
<td>15.4</td>
<td>7.57</td>
<td>15.5</td>
</tr>
<tr>
<td>Critical gel</td>
<td>(\phi = 1)</td>
<td>(\tilde{\lambda} = 0.406)</td>
<td>(n = 0.259)</td>
<td>99.9</td>
<td>10.8</td>
</tr>
<tr>
<td>Range of (\omega^*)</td>
<td>(\tilde{\lambda} = 0.246)</td>
<td>(n = 0.118)</td>
<td>33.1</td>
<td>17.7</td>
<td>42.5</td>
</tr>
</tbody>
</table>

4 Conclusions

In this work, we have outlined a generalized mathematical framework for the design of linear viscoelastic materials and structures. Importantly, frequency-dependent viscoelastic moduli (e.g., \(G', G''\) in shear) are not appropriate design parameters in general, due to the Kramers–Kronig constraint (Eq. (7)). Instead, a single kernel function, such as a stress-relaxation modulus \(G(t)\) or force-relaxation stiffness \(K(t)\), is the design-appropriate function to optimize. Driven by performance objectives, the paradigm identifies optimal target properties, in the form of a function-valued curve. The target function is agnostic to the underlying material structure or spring-dashpot arrangement. This improves achievable performance of mechanical systems by expanding the design space.

To simplify the underlying mathematics of the problem, common parameterizations for viscoelastic fluid models were chosen in place of a fully generalized relaxation kernel, \(K(t)\): linear viscous damper, Maxwell fluid model, and critical gel model. These models represent practical descriptions of many rheologically complex fluids. Additionally, the parameterization allows for the use of well-established optimization tools to demonstrate this framework.

A case study of a simple vibration isolator demonstrated the utility of parameterizations—Maxwell and critical gel models, specifically—of a generalized linear viscoelastic connection to improve performance of the system compared to the best performance available through linear dashpots. In this case, a single-mode Maxwell model proved effective for a structure and loading scenario with a single timescale. Other models, such as the critical gel and multimode Maxwell, may show even further performance improvement for more complex forcing scenarios or structures with a broad spectrum of inherent time scales.

An additional case study involved systems covering a range of possible natural frequencies, but with an unchanging viscoelastic connection to isolate vibrations. Again, a single-mode Maxwell model showed the largest improvement over the conventional linear dashpot system.

This work is limited to early stages in the design hierarchy (Fig. 1, top half) relating performance (in this case, the effectiveness of a vibration isolation system) to material properties (the generalized viscoelastic kernel function, \(K(t)\)). Later steps in the design process (bottom half of Fig. 1, which are not detailed here) include the design and fabrication of a real viscoelastic material system to achieve the target \(K(t)\) of the engineering system found in these early steps. Although difficult, this is known to some extent, as the characteristic relaxation time for various viscoelastic materials (i.e., emulsions, polymer solutions and melts, colloidal suspensions and gels) follow known scaling laws [60–62]. Specifically, it is possible to design the viscoelastic properties of certain polymer systems by controlling the molecular weight [61] or crosslinking type and amount [27].

In later stages of the design process, such as the design and fabrication of actual materials, the exact design target may not be
precisely achievable; it is not yet known how sensitive the performance of a system will be to the shape of the kernel function. This sensitivity analysis stands as important future work. An alternative approach is to identify an appropriate material class based on the optimal target $K(t)$ functions, and reformulate the system optimization problem with additional constraints due to material-specific properties. This would result in an optimal solution that is informed by the early-stage material design results, and is closer to a physically realizable material specification.

The methodology outlined in this work can be further generalized to eliminate the need to specify a functional form and parameterization of the linear viscoelastic kernel [63]. This will bring the challenge of optimizing a high-dimensional object, but would allow for a greater expansion of the early stages of the design space, while only using a single time-dependent material function as a system-level variable. This type of study is particularly relevant to early-stage design investigations before many design decisions are finalized and activities are more exploratory in nature. The results here serve as a foundation for design with rheological complexity, including efforts to expand beyond linear viscoelasticity to a fully nonlinear viscoelastic behavior and complex fluids which may provide a wide range of novel functionality yet to be discovered.

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References


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